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Valuation and hedging of European contingent claims on power with spikes: a non-Markovian approach

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Abstract. A new approach to modeling spikes in power prices proposed earlier by the author is presented and further developed. In contrast to the standard approaches, power prices with spikes as a non-Markovian stochastic process are modeled that allows for modeling spikes directly as self-reversing jumps. It is shown how this approach can be used to value and hedge European contingent claims on power with spikes. It is also shown that the values of European contingent claims on power with spikes satisfy the Cauchy problem for a certain linear evolution equation. In this way, the values of European contingent claims on power with spikes can be represented in terms of the Green's function for this Cauchy problem and the Green's function itself can be interpreted in terms of the values of the Arrow-Debreu securities on power with spikes.

Key words: contingent claims on power, non-Markovian price processes, power prices with spikes

1. Introduction

As the power markets are becoming deregulated worldwide, the modeling of spikes in power prices is becoming a key problem in risk management, physical assets valuation, and derivatives pricing.

In this article we present and further develop a new approach to modeling spikes in power prices proposed by the author in [1–4]. The main motivation for our approach is that, in our opinion, different mechanisms should be responsible for the reversion of power prices to their long-term mean between spikes and for the reversion of power prices to their long-term mean during spikes, that is, for the decay of spikes. This is due to the substantial difference in the scales of the deviations of power prices from their long-term mean between spikes and during spikes. For example, power prices in the United States Midwest in June 1998 rose to $7,500 per megawatt hour (MWh) compared with typical prices of around $30/MWh as a result of unseasonably hot weather, planned and unplanned outages, and transmission constrains [5].

In contrast to the standard approaches (see, for example, [5–8]) we model power prices with spikes as a non-Markovian stochastic process that allows for modeling spikes directly as self-reversing jumps. In this way, different mechanisms are, in fact, responsible for the reversion of power prices to their long-term mean between spikes and during spikes.

Moreover, we suggest that employing a Markov process to model power prices with spikes is ultimately making the same mechanism responsible for the reversion of power prices to their long-term mean both between spikes and during spikes. Indeed, although a Markov process can produce a sharp upward price movement as a suitable jump, it can not remember the magnitude of this sharp upward price movement to separately produce a shortly followed sharp downward price movement of approximately the same magnitude so that a spike can form.

While the stochastic process we propose is non-Markovian, it can be represented as a product of two Markov processes, the spike process and the process for power prices in the absence of spikes. The spike process is responsible for modeling spikes in power prices. It is either equal to a multiplicative magnitude of a spike during the spike or to unity between spikes. The spike process is constructed with the help of a two-state Markov process in continuous time. This two-state Markov process determines whether power prices are in the spike state, that is, during a spike or in the inter-spike state, that is, between spikes. If power prices are in the spike state, the multiplicative magnitude of the spike is determined by a random variable with a suitable distribution. We note that the spike process constructed in this way, among other things, allows for the parameters of spikes such as duration, frequency and magnitude to be relatively easily estimated directly from market data. The remaining process, the process for power prices in the absence of spikes is responsible for modeling power prices between spikes.

Although the process we propose is non-Markovian, it can be represented as a Markov process with a suitably extended state space that, in addition to the power price, also includes the magnitude of spikes. With the help of this Markov process with the extended state space we obtain a relatively simple analytical expression for the values of European contingent claims on power with spikes provided that the corresponding European contingent claims can be valued in the case of the process for power prices in the absence of spikes. This, for example, enables us to obtain a relatively simple analytical expression for power forward prices for power with spikes provided that corresponding power forward prices are known in the case of the process for power prices in the absence of spikes. Moreover, with the help of the Markov process with the extended state space, we show that the values of European contingent claims on power with spikes satisfy the Cauchy problem for a certain linear evolution equation. In this way, the values of European contingent claims on power with spikes can be represented in terms of the Green's function for this Cauchy problem and the Green's function itself can be interpreted in terms of the values of the Arrow-Debreu securities on power with spikes.

Another outcome of the non-Markovian process we propose is that it provides a natural mechanism to explain the absence of spikes in the values of European contingent claims on power far enough from their expiration time while power prices exhibit spikes. As a special case we show that power forward prices far enough from the maturity time of the forward contracts on power do not exhibit spikes while power prices, or more precisely power spot prices, do.

Finally, we consider a practically important special case when power forward prices for power without spikes follow a geometric Brownian motion. This is, for example, when power prices in the absence of spikes follow a geometric mean-reverting process. We show that in this case power forward prices for power with spikes also follow the same geometric Brownian motion far enough from the maturity time of the forward contracts on power. This allows for the valuation of European contingent claims on forwards on power for power with spikes in terms of European contingent claims on forwards on power in the Black-Scholes market environment. This also allows for the dynamic hedging of European contingent claims on forwards on power for power with spikes via the standard delta hedging.

2. The non-Markovian process for power prices with spikes

In this section we define the non-Markovian process for power prices with spikes as a product of two Markov processes, the spike process and the process for power prices in the absence of spikes. The spike process is responsible for modeling spikes in power prices and the process for power prices in the absence of spikes is responsible for modeling power prices between spikes.

2.1. THE TWO-STATE MARKOV PROCESS

As we discussed in the Introduction, we construct the spike process with the help of a two-state Markov process in continuous time. This two-state Markov process will determine whether power prices are in the *spike state*, that is, during a spike or in the *inter-spike state*, that is, between spikes.

Denote this two-state Markov process with continuous time $t \geq 0$ by M_t. Let

$$P(T,t) = \begin{pmatrix} P_{ss}(T,t) & P_{sr}(T,t) \\ P_{rs}(T,t) & P_{rr}(T,t) \end{pmatrix}, \ t \leq T,$$

be its 2×2 transition matrix, where $P_{ss}(T,t)$ and $P_{rs}(T,t)$ are the transition probabilities from the spike state at time t to the spike and inter-spike states at time T, and where $P_{sr}(T,t)$ and $P_{rr}(T,t)$ are the transition probabilities from the inter-spike state at time t to the spike and inter-spike states at time T. We comment that the subscripts s and r stand for the spike state and inter-spike, or regular, state.

We note that the Kolmogorov-Chapman equation for the Markov process M_t can be represented as follows

$$P(T,t) = P(T,\tau)P(\tau,t), \ t \leq \tau \leq T, \tag{1}$$

where $P(T,t)$ is the 2×2 identity matrix whenever $t = T$.

The Markov process M_t can also be characterized in terms of its generators, that is, 2×2 matrixes defined as the rate of change of the transition matrix $P(T,t)$ at times $t \geq 0$. It turns out that it is these generators of the Markov process M_t that will determine the duration and frequency of spikes, that is, the expected lifetime of spikes and expected time between spikes. In this way, it is these generators of the Markov process M_t that can be estimated directly from market data.

More precisely, a one-parameter family $\{L(t) : t \geq 0\}$ of 2×2 real matrixes defined by

$$L(t) = \frac{\mathrm{d}}{\mathrm{d}T} \ P(T,t)|_{T=t} \tag{2}$$

is said to *generate* the Markov process M_t and the matrix

$$L(t) = \begin{pmatrix} L_{ss}(t) & L_{sr}(t) \\ L_{rs}(t) & L_{rr}(t) \end{pmatrix}$$

is called a *generator*.

It can be shown that for each time $t \geq 0$:

$$L_{ss}(t) + L_{rs}(t) = 0 \text{ and } L_{ss}(t) \leq 0, \ \mathrm{L}_{rs}(t) \geq 0,$$
$$L_{rr}(t) + L_{sr}(t) = 0 \text{ and } \mathrm{L}_{rr}(t) \leq 0, \ \mathrm{L}_{sr}(t) \geq 0,$$

and hence each generator $L(t)$ can be characterized by two non-negative real numbers

$$a(t) = -L_{ss}(t) = L_{rs}(t) \text{ and } b(t) = -L_{rr}(t) = L_{sr}(t)\ . \tag{3}$$

In terms of the generators of the Markov process M_t, each transition matrix $P(T,t)$ is given by the following exponential of a matrix

$$P(T,t) = \mathrm{e}^{\int_t^T L(\tau)\mathrm{d}\tau}. \tag{4}$$

We comment that in the case when the generators $L(t)$ do not commute for different times $t \geq 0$, the exponential of the matrix in (4) has to be replaced by the product integral. (For the definition of the product integral see, for example, [9, Chapter 1]).

In the practically important special case of a time-homogeneous Markov process M_t, the transition matrix $P(T,t)$ is, in fact, a function of the difference $T-t$:

$$P(T-t) = \begin{pmatrix} P_{ss}(T-t) & P_{sr}(T-t) \\ P_{rs}(T-t) & P_{rr}(T-t) \end{pmatrix}.$$

In this case, due to relationship (2), the generators $L(t)$ are, in fact, time-independent:

$$L = \begin{pmatrix} L_{ss} & L_{sr} \\ L_{rs} & L_{rr} \end{pmatrix},$$

and hence relationship (4) takes the following form $P(T-t) = \mathrm{e}^{(T-t)L}$, $t \leq T$. Therefore, in the case of a time-homogeneous Markov process M_t with the generator L the transition matrix $P(T-t)$ is given by

$$P(T-t) = \begin{pmatrix} \dfrac{b + a\mathrm{e}^{-(T-t)(a+b)}}{a+b} & \dfrac{b - b\mathrm{e}^{-(T-t)(a+b)}}{a+b} \\ \dfrac{a - a\mathrm{e}^{-(T-t)(a+b)}}{a+b} & \dfrac{a + b\mathrm{e}^{-(T-t)(a+b)}}{a+b} \end{pmatrix}, \tag{5}$$

where, according to relationship (3), $a \geq 0$ and $b \geq 0$ are given by $a = -L_{ss} = L_{rs}$ and $b = -L_{rr} = L_{sr}$, and where the case of $a + b = 0$ is understood as the corresponding limit.

In another practically important special case when the Markov process M_t is time-homogeneous on time subintervals, its generators $L(t)$ are time-independent on these time subintervals and hence the transition probabilities $P(T,t)$ can also be found analytically with the help of relationships (1) and (5). In practice, a general two-state Markov process M_t can be approximated with any desired degree of accuracy by a two-state Markov process that is time-homogeneous on time subintervals.

Later in the article we will need the following decompositions of the transition probabilities of the Markov process M_t (see, for example, [1–4]):

$$\begin{aligned} P_{ss}(T,t) &= \mathrm{e}^{\int_t^T L_{ss}(\tau)\mathrm{d}\tau} + \int_t^T P_{rs}(\tau,t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'}\mathrm{d}\tau, \\ P_{sr}(T,t) &= \int_t^T P_{rr}(\tau,t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'}\mathrm{d}\tau. \end{aligned} \tag{6}$$

We note that the first relationship in (6) can also be written as follows:

$$P_{ss}(T,t) = P_{ss}^s(T,t) + P_{ss}^r(T,t), \tag{7}$$

where

$$P_{ss}^s(T,t) = \mathrm{e}^{\int_t^T L_{ss}(\tau)\mathrm{d}\tau} \tag{8}$$

and

$$P_{ss}^{r}(T,t) = \int_{t}^{T} P_{rs}(\tau,t) L_{sr}(\tau) \mathrm{e}^{\int_{\tau}^{T} L_{ss}(\tau')\mathrm{d}\tau'} \mathrm{d}\tau. \tag{9}$$

2.2. The spike process

We will define the spike process that will be responsible for modeling spikes in power prices. Let $\xi_t > 1$ with $t \geq 0$ be independent random variables with the probability density functions $\Xi(t,\xi)$. We comment that ξ_t will be interpreted as the multiplicative magnitude of spikes that start at time t and $\Xi(t,\xi)$ as the conditional probability density function for the mutiplicative magnitude of spikes that start at time t.

We define the spike process $\lambda_t \geq 1$ with $t \geq 0$ as follows. If the Markov process M_t is in the inter-spike state then the spike process λ_t is equal to unity. If the Markov process M_t transits into the spike state at time τ, then the spike process λ_t is equal to the value of the random variable ξ_τ during the entire time that the Markov process M_t remains in the spike state. Finally, if the Markov process M_t is in the spike state at time $t = 0$ then the spike process λ_t is initiated at some $\lambda_0 > 1$.

It can be shown [1–4] that the spike process λ_t is, in fact, a Markov process and that its transition probability density function $\Lambda(t,T,\lambda_t,\lambda_T)$, with the help of relationships in (6), is given by

$$\Lambda(t,T,\lambda_t,\lambda_T) = \begin{cases} \begin{array}{l} \mathrm{e}^{\int_t^T L_{ss}(\tau)\mathrm{d}\tau}\delta(\lambda_t - \lambda_T) + \\ \int_t^T \Xi(\tau,\lambda_T) P_{rs}(\tau,t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'}\mathrm{d}\tau + \\ P_{rs}(T,t)\delta(1-\lambda_T) \end{array} & \text{if } \lambda_t > 1 \\ \begin{array}{l} \int_t^T \Xi(\tau,\lambda_T) P_{rr}(\tau,t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'}\mathrm{d}\tau + \\ P_{rr}(T,t)\delta(1-\lambda_T) \end{array} & \text{if } \lambda_t = 1 \end{cases}, \tag{10}$$

where $\delta(x)$ is the Dirac delta function.

Finally, we will say that the spike process λ_t is in the *spike state* or *inter-spike state* if the Markov process M_t is in the spike state or inter-spike state.

2.3. The process for power prices in the absence of spikes

We will define the process for power prices in the absence of spikes that will be responsible for modeling power prices between spikes. Denote by $\hat{\Psi}_t > 0$ the price (for example, in dollars) of a unit of power (for example, in MWh) at time $t \geq 0$ in the absence of spikes in power prices. Assume that $\hat{\Psi}_t$ can be represented as a Markov process. For example, $\hat{\Psi}_t$ can be a diffusion process defined by the following stochastic differential equation

$$\mathrm{d}\hat{\Psi}_t = \mu(\hat{\Psi}_t,t)\mathrm{d}t + \sigma(\hat{\Psi}_t,t)\mathrm{d}W_t, \tag{11}$$

where $\mu(\hat{\Psi}_t,t)$ is the drift, $\sigma(\hat{\Psi}_t,t) > 0$ is the volatility, and W_t is the Wiener process. In the practically important special case of the geometric mean-reverting process the preceding stochastic differential equation takes the following form

$$\mathrm{d}\hat{\Psi}_t = \eta(t)(\mu(t) - \log\hat{\Psi}_t)\hat{\Psi}_t\mathrm{d}t + \sigma(t)\hat{\Psi}_t\mathrm{d}W_t, \tag{12}$$

where $\eta(t) > 0$ is the mean-reversion rate, $\mu(t)$ is the equilibrium mean, and $\sigma(t) > 0$ is the volatility.

2.4. THE NON-MARKOVIAN PROCESS FOR POWER PRICES WITH SPIKES

Now we are ready to define the non-Markovian process for power prices with spikes. Denote by $\Psi_t > 0$ the price (for example, in dollars) of a unit of power (for example, in MWh) at time $t \geq 0$ in the presence of spikes in power prices. Assume that the spike process λ_t and the process $\hat{\Psi}_t$ for power prices in the absence of spikes are independent. Define the process Ψ_t with $t \geq 0$ as the product of the spike process λ_t and the process $\hat{\Psi}_t$ for power prices in the absence of spikes:

$$\Psi_t = \lambda_t \hat{\Psi}_t. \tag{13}$$

It is easy to see that Ψ_t is a non-Markovian process. Indeed, in order to characterize the future behavior of the process Ψ_t we need to know the current value of either the process λ_t or the process $\hat{\Psi}_t$ in addition to the current value of the process Ψ_t. In other words, the state of the power market at any time $t \geq 0$ can be fully characterized by any pair of the values of the processes Ψ_t, λ_t, and $\hat{\Psi}_t$ at time t. Moreover, it can be shown [2] that, although the process Ψ_t is non-Markovian, it can be, in fact, represented as a Markov process that for any time $t \geq 0$ can be fully characterized, for example, by the values of the processes λ_t and $\hat{\Psi}_t$ at time t. Equivalently, the non-Markovian process Ψ_t can be represented as a Markov process with the extended state space that at any time $t \geq 0$ consists of all possible pairs $(\lambda_t, \hat{\Psi}_t)$ with $\lambda_t \geq 1$ and $\hat{\Psi}_t > 0$. We will use this representation of the non-Markovian process Ψ_t as a Markov process with the extended state space later in the article in the valuation of European contingent claims on power with spikes.

We will say that the process Ψ_t for power prices with spikes is in the *spike state* or *inter-spike state* if the spike process λ_t is in the spike state or inter-spike state, or equivalently, if the Markov process M_t is in the spike state or inter-spike state.

Now we will show that the spike state of the process Ψ_t can mimic spikes in power prices, that is, exhibit sharp upward price movements shortly followed by equally sharp downward price movements of approximately the same magnitude. Since the expected times $\bar{t}_s$ and $\bar{t}_r$ for the process Ψ_t to be in the spike and inter-spike states starting at time t coincide with those of the Markov process M_t, we have

$$\begin{aligned} \bar{t}_s &= \int_t^\infty (\tau - t) \mathrm{e}^{-\int_t^\tau a(\tau')\mathrm{d}\tau'} a(\tau)\mathrm{d}\tau, \\ \bar{t}_r &= \int_t^\infty (\tau - t) \mathrm{e}^{-\int_t^\tau b(\tau')\mathrm{d}\tau'} b(\tau)\mathrm{d}\tau, \end{aligned} \tag{14}$$

where $a(t)$ and $b(t)$ are defined in relationship (3).

In a practically important special case of a time-homogeneous Markov process M_t relationships in (14) take the following form

$$\begin{aligned} \bar{t}_s &= \int_t^\infty (\tau - t) \mathrm{e}^{-a(\tau - t)} a\mathrm{d}\tau = \frac{1}{a}, \\ \bar{t}_r &= \int_t^\infty (\tau - t) \mathrm{e}^{-b(\tau - t)} b\mathrm{d}\tau = \frac{1}{b}, \end{aligned} \tag{15}$$

so that the expected times $\bar{t}_s$ and $\bar{t}_r$ for the process Ψ_t to be in the spike and inter-spike states starting at time t do not depend on t. If for each time $t \geq 0$ the expected time $\bar{t}_s$ for the process

Ψ_t to be in the spike state is small relative to a characteristic time of change of the process $\hat{\Psi}_t$, the spike state of the process Ψ_t can be interpreted as a spike in power prices. For example, if $\hat{\Psi}_t$ is a diffusion process defined in (11) then the condition mentioned above can be restated as follows

$$\sigma^2(\hat{\Psi}_t, t)\, \bar{t}_s \ll 1 \quad \text{and} \quad \mu(\hat{\Psi}_t, t)\, \bar{t}_s \ll 1, \quad t \geq 0,$$

where $\mu(\hat{\Psi}_t, t)$ is the drift and $\sigma(\hat{\Psi}_t, t)$ is the volatility of $\hat{\Psi}_t$.

Under this interpretation, $\bar{t}_s$ is the expected lifetime of a spike that starts at time t, and $\bar{t}_r$ is the expected time between two consecutive spikes when the first spike ends at time t. In this way, $a(t)$ and $b(t)$ control the duration and frequency of spikes by means of relationships in (14). For example, it is easy to see with the help of relationships in (14) that for short-lived spikes $a(t)$ has to be relatively large, while for rare spikes $b(t)$ has to be relatively small.

The spike process λ_t is now responsible for modeling spikes in power prices. It is either equal to a multiplicative magnitude of a spike during the spike or to unity between spikes. In this way, $\Xi(t, \lambda)$ can be interpreted as the conditional probability density function for the multiplicative magnitude of spikes that start at time t. Moreover, the Markov process M_t determines whether power prices are in the spike state, that is, during a spike or in the inter-spike state, that is, between spikes. Finally, since the spike process λ_t is equal to unity between spikes, the process Ψ_t for power prices with spikes by its definition in relationship (13) coincides between spikes with the process $\hat{\Psi}_t$ for power prices in the absences of spikes. Therefore, the process $\hat{\Psi}_t$ for power prices in the absence of spikes is responsible for modeling power prices between spikes.

We note that the parameters related to spikes are relatively easy to estimate directly from market data. Indeed, under the assumption that the Markov process M_t is time-homogeneous, according to relationships in (15) the parameters a and b can be estimated as the inverses of the average lifetime of spikes and the average time between spikes. In practice, as we pointed out earlier in the article, a general two-state Markov process M_t can be approximated with any desired degree of accuracy by a two-state Markov process that is time-homogeneous on time subintervals. The estimation of the conditional probability density function $\Xi(t, \lambda)$ for the multiplicative magnitude of spikes can be based on the standard parametric or nonparametric statistical methods. For the parametric statistical methods scaling or asymptotically scaling probability distributions are of a particular practical importance [1–3].

Finally, we point out that the process Ψ_t for power prices with spikes can also be interpreted in terms of a structural model [2]. For example, the Markov process M_t can be interpreted as a service-repair process of a representative power plant whose failure at time $t \geq 0$ results in the spike in power prices with the multiplicative magnitude $\lambda_t > 1$ determined by the conditional probability-density function $\Xi(t, \lambda)$. In this way, the spike state and inter-spike state of the Markov process M_t can be interpreted as the down state, that is, the state when the power plant is not in service while being repaired and the up state, that is, the state when the power plant is in service. Moreover, $a(t)$ and $b(t)$ that determine the generators of the Markov process M_t via relationships in (3) can be interpreted as the repair rate and failure rate of the power plant at time t. In turn, the expected lifetime $\bar{t}_s$ of a spike that starts at time t and the expected time $\bar{t}_r$ between two consecutive spikes when the first spike ends at time t can be interpreted as the mean down time, that is, mean time to repair and the mean up time, that is, mean time to failure of the power plant at time t.

3. European contingent claims on power with spikes

In this section we present an analytical expression for the values of European contingent claims on power in the case when the dynamics of the power prices with spikes are described by the non-Markovian process Ψ_t defined earlier in the article. This is assuming that the corresponding European contingent claims can be valued in the case of the process $\hat{\Psi}_t$ for power prices in the absence of spikes. We also show how the values of European contingent claims on power far enough from their expiration time do not exhibit spikes while the power prices do.

3.1. Valuation of European contingent claims on power without spikes

In order to value European contingent claims on power in the case of the power prices with spikes following the non-Markovian process Ψ_t, we need to value the corresponding European contingent claims in the case of the process $\hat{\Psi}_t$ for power prices in the absence of spikes.

For a European contingent claim with inception time t, expiration time T and payoff g, denote by $\hat{E}(t, T, g) = \hat{E}(t, T, g)(\hat{\Psi}_t)$ its value at the inception time t as a function of the power price $\hat{\Psi}_t$ at time t. We comment that, in contrast to a European option whose payoff is always nonegative, the payoff of a European contingent claim, as a function of the power price, can take both positive and negative values.

Assume now that $\hat{\Psi}_t$ is a risk-neutral process for the power prices in the absence of spikes. Then the value $\hat{E}(t, T, g) = \hat{E}(t, T, g)(\hat{\Psi}_t)$ of the European contingent claim with inception time t, expiration time T and payoff g can be found as the discounted expected value of its payoff

$$\hat{E}(t, T, g)(\hat{\Psi}_t) = \mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau} \int_0^\infty P(t, T, \hat{\Psi}_t, \hat{\Psi}_T) g(\hat{\Psi}_T)\mathrm{d}\hat{\Psi}_T, \tag{16}$$

where $r(t)$ is the continuously compounded interest rate, and $P(t, T, \hat{\Psi}_t, \hat{\Psi}_T)$ is the risk-neutral transition probability density function for $\hat{\Psi}_t$ as a Markov process.

For example, in the practically important special case of the mean-reverting market environment, that is, when $\hat{\Psi}_t$ follows a geometric mean-reverting process defined in (12), relationship (16) takes the following form [10], [11]:

$$\hat{E}^{MR}(t, T, g)(\hat{\Psi}_t) = \frac{\mathrm{e}^{-\bar{r}(t,T)(T-t)}}{\hat{\sigma}(t, T)\sqrt{2\pi(T-t)}} \int_0^\infty \mathrm{e}^{-\frac{1}{2}\frac{(a(t,T)\log\hat{\Psi}_t + b(t,T) - \log\hat{\Psi}_T)^2}{\hat{\sigma}^2(t,T)(T-t)}} g(\hat{\Psi}_T)\frac{\mathrm{d}\hat{\Psi}_T}{\hat{\Psi}_T}, \tag{17}$$

where:

$$\bar{r}(t, T) = \frac{1}{T-t}\int_t^T r(\tau)\mathrm{d}\tau,$$

$$\hat{\sigma}(t, T) = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)\mathrm{e}^{-2\int_\tau^T \eta(\tau')\mathrm{d}\tau'}\mathrm{d}\tau},$$

$$a(t, T) = \mathrm{e}^{-\int_t^T \eta(\tau)\mathrm{d}\tau},$$

$$b(t, T) = \int_t^T \eta(\tau)\left(\mu(\tau) - \frac{1}{2}\frac{\sigma^2(\tau)}{\eta(\tau)}\right)\mathrm{e}^{-\int_\tau^T \eta(\tau')\mathrm{d}\tau'}\mathrm{d}\tau.$$

Later in the article we will need an expression for the value of a European contingent claim in the mean-reverting market environment in terms the value of the corresponding

European contingent claim in the Black-Scholes market environment, that is, when the risk-neutral dynamics of the price S_t of an underlying security are given by the geometric Brownian motion

$$dS_t = \mu_{BS}(t)S_t dt + \sigma_{BS}(t)S_t dW_t,$$

where $\mu_{BS}(t)$ is the drift and $\sigma_{BS}(t) > 0$ is the volatility.

The value $\hat{E}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,g) = \hat{E}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,g)(S_t)$ in the Black-Scholes market environment of a European contingent claim with inception time t, expiration time T and payoff g is given by (see, for example, [12, Chapters 8, 10]):

$$\hat{E}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,g)(S_t) = \frac{e^{-\bar{r}(t,T)(T-t)}}{\bar{\sigma}_{BS}\sqrt{2\pi(T-t)}} \int_0^\infty e^{-\frac{1}{2}\frac{(\log(S_t/S_T)+(\bar{\mu}_{BS}-\frac{1}{2}\bar{\sigma}^2_{BS})(T-t))^2}{\bar{\sigma}^2_{BS}(T-t)}} g(S_T)\frac{dS_T}{S_T}, \quad (18)$$

where

$$\bar{\sigma}_{BS} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2_{BS}(\tau)d\tau},$$
$$\bar{\mu}_{BS} = \frac{1}{T-t}\int_t^T \mu_{BS}(\tau)d\tau.$$

It can be shown [10], [13] that, with the help of relationships (17) and (18), the value $\hat{E}^{MR}(t,T,g)$ can be expressed in terms of the value $\hat{E}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,g)$ as follows:

$$\hat{E}^{MR}(t,T,g)(\hat{\Psi}_t) = \hat{E}^{BS}_{\hat{\sigma}(t,T),0}(t,T,g)(\hat{\Psi}_t^{a(t,T)} e^{b(t,T)} e^{(T-t)\frac{1}{2}\hat{\sigma}^2(t,T)}). \quad (19)$$

In this way, the values $\hat{C}^{MR}(t,T,\hat{\Psi}_t,X)$ and $\hat{P}^{MR}(t,T,\hat{\Psi}_t,X)$ in the mean-reverting market environment of the European call and put options with inception time t, expiration time T and strike X can be expressed [10], [13] in terms of the values $\hat{C}^{BS}_{\hat{\sigma}(t,T),0}(t,T,S_t,X)$ and $\hat{P}^{BS}_{\hat{\sigma}(t,T),0}(t,T,S_t,X)$ of the corresponding European call and put options in the Black-Scholes market environment:

$$\begin{aligned} \hat{C}^{MR}(t,T,\hat{\Psi}_t,X) &= \hat{C}^{BS}_{\hat{\sigma}(t,T),0}(t,T,\hat{\Psi}_t^{a(t,T)} e^{b(t,T)} e^{(T-t)\frac{1}{2}\hat{\sigma}^2(t,T)},X), \\ \hat{P}^{MR}(t,T,\hat{\Psi}_t,X) &= \hat{P}^{BS}_{\hat{\sigma}(t,T),0}(t,T,\hat{\Psi}_t^{a(t,T)} e^{b(t,T)} e^{(T-t)\frac{1}{2}\hat{\sigma}^2(t,T)},X), \end{aligned} \quad (20)$$

where $\hat{C}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,S_t,X)$ and $\hat{P}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,S_t,X)$ are given by the Black-Scholes formulas (see, for example, [12, Chapters 7, 8]):

$$\begin{aligned} \hat{C}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,S_t,X) &= S_t e^{(\bar{\mu}_{BS}-\bar{r}(t,T))(T-t)}N(d_+) - Xe^{-\bar{r}(t,T)(T-t)}N(d_-), \\ \hat{P}^{BS}_{\bar{\sigma}_{BS},\bar{\mu}_{BS}}(t,T,S_t,X) &= Xe^{-\bar{r}(t,T)(T-t)}N(-d_-) - S_t e^{(\bar{\mu}_{BS}-\bar{r}(t,T))(T-t)}N(-d_+), \end{aligned} \quad (21)$$

with

$$d_\pm = \frac{\log(S_t/X) + (\bar{\mu}_{BS} \pm \frac{1}{2}\bar{\sigma}^2_{BS})(T-t)}{\bar{\sigma}_{BS}\sqrt{(T-t}},$$
$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-y^2/2}dy.$$

3.2. Valuation of European contingent claims on power with spikes

We will present an analytical expression for the values of European contingent claims on power in the case when the dynamics of the power prices with spikes are given by the non-Markovian process Ψ_t.

Consider a European contingent claim on power with inception time t, expiration time T and payoff g that explicitly depends, in addition to the power price Ψ_T at time T, on the state, spike or inter-spike state, of the power price Ψ_T and the magnitude of the related spike at time T as determined by the value of the spike process λ_T at time T. Such payoff g, in view of the definition of the process Ψ_t in (13), can be represented as a function of any pair of the values of the processes Ψ_T, λ_T, and $\hat{\Psi}_T$ at time T. Throughout this article we will represent payoff g as a function of λ_T and $\hat{\Psi}_T$, that is $g = g_{\lambda_T}(\hat{\Psi}_T) = g(\lambda_T\hat{\Psi}_T, \lambda_T)$. In this way, if the payoff g depends only on the power price $\Psi_T = \lambda_T\hat{\Psi}_T$ at time T and does not explicitly depend on the state, spike or inter-spike state, of the power price $\Psi_T = \lambda_T\hat{\Psi}_T$ at time T as determined by the value of the spike process λ_T at time T then

$$g_{\lambda_T}(\hat{\Psi}_T) = g(\lambda_T\hat{\Psi}_T). \tag{22}$$

We comment that a holder of the European contingent claim on power with inception time t, expiration time T and payoff g will receive a (positive or negative) payment of either $g_{\lambda_T}(\hat{\Psi}_T) = g(\lambda_T\hat{\Psi}_T, \lambda_T)$ units of account if the power price $\Psi_T = \lambda_T\hat{\Psi}_T$ is in the spike state at time T and the magnitude of the spike is $\lambda_T > 1$, or $g_{\lambda_T=1}(\hat{\Psi}_T) = g(\lambda_T\hat{\Psi}_T, \lambda_T)|_{\lambda_T=1}$ units of account if the power price $\Psi_T = \lambda_T\hat{\Psi}_T$ is in the inter-spike state at time T, that is $\lambda_T = 1$. We also comment that by allowing the payoffs to explicitly depend on the state, spike or inter-spike state and the magnitude of the related spike one can construct European contingent claims on power that provide a better tailored protection for their holders against spikes in power prices. Moreover, with the help of the beliefs-preferences gauge symmetry introduced by the author in [14–17] one can construct a dynamically spanning set of European contingent claims with such payoffs that allows for the beliefs-preferences-independent valuation and dynamic replication of European contingent claims on power with spikes. In turn, this allows for the beliefs-preferences-independent valuation and dynamic replication of more general contingent claims on power with spikes such as universal contingent claims introduced by the author in [10, 18–21], [2], [16], [22]. Those include Bermudan and American options.

As we discussed earlier in the article, the state of the power market at any time $t \geq 0$ can be fully characterized by any pair of the values of the processes Ψ_t, λ_t and $\hat{\Psi}_t$ at time t. In this regard, the value $E(t, T, g)$ at inception time t of a European contingent claim (on power with spikes) with inception time t, expiration time T and payoff g can be represented as a function of any pair of the values of the processes Ψ_t, λ_t,and $\hat{\Psi}_t$ at time t. Throughout this article we will represent the value $E(t, T, g)$ as a function of λ_t and $\hat{\Psi}_t$, that is $E(t, T, g) = E_{\lambda_t}(t, T, g)(\hat{\Psi}_t) = E(t, T, g)(\lambda_t\hat{\Psi}_t, \lambda_t)$. We comment that the value of the European contingent claim (on power with spikes) with inception time t, expiration time T and payoff g is either $E_{\lambda_t}(t, T, g)(\hat{\Psi}_t) = E(t, T, g)(\lambda_t\hat{\Psi}_t, \lambda_t)$ units of account if the power price $\Psi_t = \lambda_t\hat{\Psi}_t$ is in the spike state at time t and the magnitude of the spike is $\lambda_t > 1$, or $E_{\lambda_t=1}(t, T, g)(\hat{\Psi}_t) = E(t, T, g)(\lambda_t\hat{\Psi}_t, \lambda_t)|_{\lambda_t=1}$ units of account if the power price $\Psi_t = \lambda_t\hat{\Psi}_t$ is in the inter-spike state at time t, that is $\lambda_t = 1$.

As we also discussed earlier in the article, the non-Markovian process Ψ_t can be represented as a Markov process with the state space that at any time $t \geq 0$ consists of all possible pairs $(\lambda_t, \hat{\Psi}_t)$ with $\lambda_t \geq 1$ and $\hat{\Psi}_t > 0$. Assume now that Ψ_t is a risk-neutral process for power prices with spikes so that it can be represented as a risk-neutral Markov process with the state space that at any time $t \geq 0$ consists of all possible pairs $(\lambda_t, \hat{\Psi}_t)$ with $\lambda_t \geq 1$ and $\hat{\Psi}_t > 0$. Then the value $E(t, T, g)$ of the European contingent claim (on power with spikes) with inception time t, expiration time T and payoff g can be found as the discounted expected value of its payoff. Moreover, it can be shown [1], [2], [4] with the help of relationships (10) and (16) that the value $E(t, T, g)$ of the European contingent claim on power with spikes can be represented in terms of the value of the following European contingent claim on power without spikes:

$$\begin{aligned} E_{\lambda_t}(t, T, g)(\hat{\Psi}_t) &= \mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau} \int_0^\infty \int_1^\infty P(t, T, \hat{\Psi}_t, \hat{\Psi}_T)\Lambda(t, T, \lambda_t, \lambda_T) g_{\lambda_T}(\hat{\Psi}_T)\mathrm{d}\lambda_T \mathrm{d}\hat{\Psi}_T \\ &= \hat{E}(t, T, \bar{g}_{\lambda_t})(\hat{\Psi}_t), \end{aligned} \tag{23}$$

where the payoff $\bar{g}_{\lambda_t} = \bar{g}_{\lambda_t}(\hat{\Psi}_T)$ is the risk-neutral *average of the payoff* g_{λ_T} *over the magnitude of spikes* λ_T at time T given by

$$\bar{g}_{\lambda_t} = \begin{cases} P_{ss}^s(T, t) g_{\lambda_t} + \int_1^\infty \Xi_s(t, T, \lambda_T) g_{\lambda_T} \mathrm{d}\lambda_T + P_{rs}(T, t) g_1 & \text{if } \lambda_t > 1 \\ \int_1^\infty \Xi_r(t, T, \lambda_T) g_{\lambda_T} \mathrm{d}\lambda_T + P_{rr}(T, t) g_1 & \text{if } \lambda_t = 1 \end{cases} \tag{24}$$

with

$$\begin{aligned} \Xi_s(t, T, \lambda_T) &= \int_t^T \Xi(\tau, \lambda_T) P_{rs}(\tau, t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'} \mathrm{d}\tau, \\ \Xi_r(t, T, \lambda_T) &= \int_t^T \Xi(\tau, \lambda_T) P_{rr}(\tau, t) L_{sr}(\tau) \mathrm{e}^{\int_\tau^T L_{ss}(\tau')\mathrm{d}\tau'} \mathrm{d}\tau, \end{aligned} \tag{25}$$

and where $P_{ss}^s(T, t)$ is given in (8).

We comment that, due to the linearity of the value of a European contingent claim as a function of its payoff and relationship (24), the value $E(t, T, g)$ in (3.2) can also be represented as the risk-neutral average of the value $\hat{E}(t, T, g_{\lambda_T})$ over the magnitude of spikes λ_T at time T. Since this risk-neutral average can be viewed as a linear combination, possibly continuous, of the values $\hat{E}(t, T, g_{\lambda_T})$, a European contingent claim on power with spikes can also be viewed as the corresponding portfolio of European contingent claims on power without spikes.

In a special case when the conditional probability density function $\Xi(t, \lambda)$ for the mutiplicative magnitude of spikes that start at time t is independent of time t and is equal to $\Xi(\lambda)$, relationship (24) takes the following form

$$\bar{g}_{\lambda_t} = \begin{cases} P_{ss}^s(T, t) g_{\lambda_t} + P_{ss}^r(T, t)\bar{g} + P_{rs}(T, t) g_1 & \text{if } \lambda_t > 1 \\ P_{sr}(T, t)\bar{g} + P_{rr}(T, t) g_1 & \text{if } \lambda_t = 1 \end{cases}, \tag{26}$$

where $\bar{g}$ is the risk-neutral *conditional average of the payoff* g_{λ_T} *over the magnitude of spikes* λ_T at time T given by

$$\bar{g} = \int_1^\infty \Xi(\lambda_T) g_{\lambda_T} \mathrm{d}\lambda_T, \tag{27}$$

and where $P_{ss}^r(T, t)$ is given in (9). Moreover, in a special case of spikes with constant magnitude λ, that is, when the conditional probability density function $\Xi(\lambda')$ is the Dirac delta function $\delta(\lambda - \lambda')$, relationship (26) and hence (24) take the following form

$$\bar{g}_{\lambda_t} = \begin{cases} P_{ss}(T, t) g_\lambda + P_{rs}(T, t) g_1 & \text{if } \lambda_t = \lambda \\ P_{sr}(T, t) g_\lambda + P_{rr}(T, t) g_1 & \text{if } \lambda_t = 1 \end{cases} \tag{28}$$

where we used the decomposition in (7).

3.3. Linear evolution equation for European contingent claims on power with spikes and the associated Green's functions

It can be shown [2] that the value $E(t, T, g)$ in (3.2) of a European contingent claim on power with spikes satisfies the Cauchy problem for the following linear evolution equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v + \hat{L}(t) v + \Lambda(t) v - r(t) v = 0,\ t < \mathrm{T}, \\ v(T) = g, \end{aligned} \tag{29}$$

where $\hat{L}(t)$ and $\Lambda(t)$ are the generators of the process $\hat{\Psi}_t$ for power prices in the absence of spikes and of the spike process λ_t as Markov processes. (For the definition of the generators of a general Markov process see, for example, [23, Chapter 2]).

It can also be shown [2] that the Green's function $G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T)$ for the Cauchy problem in (29) is given by

$$G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T) = \mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau} P(t, T, \hat{\Psi}_t, \hat{\Psi}_T) \Lambda(t, T, \lambda_t, \lambda_T),$$

where $P(t, T, \hat{\Psi}_t, \hat{\Psi}_T)$ and $\Lambda(t, T, \lambda_t, \lambda_T)$ are defined in (16) and (10). Financially, the Green's function $G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T)$ is the value of the Arrow-Debreu security on power with spikes, that is, the European option (on power with spikes) with inception time t, expiration time T and payoff $\delta(\hat{\Psi}_T - \hat{\Psi}'_T)\delta(\lambda_T - \lambda'_T)$, where $\delta(x)$ is the Dirac delta function. In this regard, the value $E_{\lambda_t}(t, T, g)(\hat{\Psi}_t)$ in (3.2) of the European contingent claim (on power with spikes) with inception time t, expiration time T and payoff g can be expressed in terms of the Green's function $G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T)$, and the European contingent claim on power with spikes can be viewed as the corresponding portfolio of the Arrow-Debreu securities on power with spikes.

We comment that, in a practically important special case when $\hat{\Psi}_t$ is a geometric mean-reverting process defined in (12), the generator $\hat{L}(t)$ is given by

$$\hat{L}(t) = \frac{1}{2}\sigma^2(t)\hat{\Psi}^2 \frac{\partial^2}{\partial \hat{\Psi}^2} + \eta(t)(\mu(t) - \log \hat{\Psi})\hat{\Psi}\frac{\partial}{\partial \hat{\Psi}}.$$

In this case, the Green's function $G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T)$ takes the following form

$$G(t, T, \hat{\Psi}_t, \hat{\Psi}_T, \lambda_t, \lambda_T) =$$

$$\mathrm{e}^{-\int_t^T r(\tau)\mathrm{d}\tau} \left(\frac{1}{\hat{\sigma}(t,T)\sqrt{2\pi(T-t)}} \mathrm{e}^{-\frac{1}{2}\frac{(a(t,T)\log \hat{\Psi}_t + b(t,T) - \log \hat{\Psi}_T)^2}{\hat{\sigma}^2(t,T)(T-t)}} \frac{1}{\hat{\Psi}_T} \right) \Lambda(t, T, \lambda_t, \lambda_T),$$

where $\hat{\sigma}(t, T)$, $a(t, T)$ and $b(t, T)$ are defined in (17).

We also comment that due to relationship (10) the generator $\Lambda(t)$ is a linear integral operator with the kernel:

$$\Lambda(t, \lambda_t, \lambda'_t) = \begin{cases} L_{ss}(t)\delta(\lambda_t - \lambda'_t) + L_{rs}(t)\delta(1 - \lambda'_t) & \text{if } \lambda_t > 1 \\ \Xi(t, \lambda'_t) L_{sr}(t) + L_{rr}(t)\delta(1 - \lambda'_t) & \text{if } \lambda_t = 1 \end{cases}.$$

For example, in the special case of spikes with constant magnitude λ the generator $\Lambda(t)$ can be represented as the 2×2 matrix $L^*(t)$ transposed to the generator $L(t)$ in (2) of the Markov process M_t. In turn, $v = v(t)$ and g can be represented as two-dimensional vector functions

$$v(t) = \begin{pmatrix} E_{\lambda_t=\lambda}(t, T, g) \\ E_{\lambda_t=1}(t, T, g) \end{pmatrix} \text{ and } g = \begin{pmatrix} g_{\lambda_T=\lambda} \\ g_{\lambda_T=1} \end{pmatrix}.$$

We point out that $\Lambda(t)$ represented as $L^*(t)$ can also be expressed in terms of the Pauli matrices which gives rise to an analogy between the linear evolution equation for European contingent claims on power with spikes in (29) and the Schrödinger equation for a two-component spinor describing a nonrelativistic spin 1/2 particle in an electromagnetic field. (For the definition of the Pauli matrices, the Schrödinger equation and related terminology see, for example, [24, Chapters 3, 4, 9]).

3.4. Why values of European contingent claims on power do not exhibit spikes while power prices do

We will show [1], [2], [4] that, when the time to expiration of a European contingent claim on power is relatively large with respect to the duration of spikes, the value of this European contingent claim does not exhibit spikes, while the power prices do.

For simplicity we consider a special case when the spikes have constant magnitude λ and the Markov process M_t is time-homogeneous. In this case the value $E(t, T, g)$ of the European contingent claim with inception time t, expiration time T and payoff g is determined by relationship (3.2) with the payoff $\bar{g}_{\lambda_t}$ given by (28), and the transition matrix $P(T, t) = P(T - t)$ of the Markov process M_t is given by (5).

With the help of relationship (5) the transition probabilities of the Markov process M_t can be represented as follows

$$\begin{aligned} P_{ss}(T-t) &= \pi_s + O(\mathrm{e}^{-(T-t)a}), \ P_{sr}(T-t) = \pi_s + O(\mathrm{e}^{-(T-t)a}), \\ P_{rs}(T-t) &= \pi_r + O(\mathrm{e}^{-(T-t)a}), \ P_{rr}(T-t) = \pi_r + O(\mathrm{e}^{-(T-t)a}), \end{aligned} \tag{30}$$

where

$$\pi_s = \frac{b}{a+b} \quad \text{and} \quad \pi_r = \frac{a}{a+b} \tag{31}$$

are the *ergodic probabilities* of the Markov process M_t to be in the spike and inter-spike states, and where $O(\mathrm{e}^{-(T-t)a})$ stands for a term of the order $\mathrm{e}^{-(T-t)a}$ or higher.

Therefore, in view of relationship (30) the payoffs $\bar{g}_{\lambda_t=\lambda}$ and $\bar{g}_{\lambda_t=1}$ in (28) coincide up to the terms of the order $O(\mathrm{e}^{-(T-t)a})$:

$$\bar{g}_{\lambda_t=\lambda} = \bar{g}_{\mathrm{erg}} + O(\mathrm{e}^{-(T-t)a}) \quad \text{and} \quad \bar{g}_{\lambda_t=1} = \bar{g}_{\mathrm{erg}} + O(\mathrm{e}^{-(T-t)a}), \tag{32}$$

where $\bar{g}_{\mathrm{erg}}$ is the risk-neutral *ergodic average of the payoff* g_{λ_T} *over the magnitude of spikes* λ_T at time T given by

$$\bar{g}_{\mathrm{erg}} = \pi_s g_{\lambda_T=\lambda} + \pi_r g_{\lambda_T=1}, \tag{33}$$

with the ergodic probabilities π_s and π_r given in (31).

In turn, the values $E_{\lambda_t=\lambda}(t, T, g)$ and $E_{\lambda_t=1}(t, T, g)$ determined by relationship (3.2) with the payoffs $\bar{g}_{\lambda_t=\lambda}$ and $\bar{g}_{\lambda_t=1}$ given by (32) also coincide up to the terms of the order $O(\mathrm{e}^{-(T-t)a})$ and hence can be combined into a single expression as follows:

$$E(t, T, g)(\hat{\Psi}_t) = \hat{E}(t, T, \bar{g}_{\mathrm{erg}})(\hat{\Psi}_t) + O(\mathrm{e}^{-(T-t)a}), \tag{34}$$

where $E(t, T, g)$ is either $E_{\lambda_t=\lambda}(t, T, g)$ or $E_{\lambda_t=1}(t, T, g)$.

In this regard, if the time to expiration $T - t$ of a European contingent claim on power with spikes is relatively large with respect to the expected lifetime of a spike $\bar{t}_s = 1/a$ then the values $E_{\lambda_t=\lambda}(t, T, g)$ and $E_{\lambda_t=1}(t, T, g)$ of the European contingent claim during a spike and between spikes differ only by an exponentially small term of the order $O(\mathrm{e}^{-(T-t)a})$. As a result, the value of a European contingent claim on power does not exhibit spikes while power prices do. In turn, the value of a European contingent claim can start exhibiting spikes when its time to expiration approaches the expected lifetime of a spike $\bar{t}_s = 1/a$.

We comment [2] that, in view of relationship (33), the value $E(t, T, g)$ in (34) can also be represented in terms of the risk-neutral ergodic average of the values $\hat{E}(t, T, g_{\lambda_T})$ over the magnitude of spikes λ_T at time T:

$$E(t, T, g)(\hat{\Psi}_t) = \pi_s \hat{E}(t, T, g_{\lambda_T=\lambda})(\hat{\Psi}_t) + \pi_r \hat{E}(t, T, g_{\lambda_T=1})(\hat{\Psi}_t) + O(\mathrm{e}^{-(T-t)a}). \tag{35}$$

For example, it can be shown [2] that, in the case of the European call and put options with strike X, relationship (35) takes the following form:

$$\begin{aligned} C(t, T, \hat{\Psi}_t, X) &= \pi_s \lambda \hat{C}(t, T, \hat{\Psi}_t, \lambda^{-1} X) + \pi_r \hat{C}(t, T, \hat{\Psi}_t, X) + O(\mathrm{e}^{-(T-t)a}), \\ P(t, T, \hat{\Psi}_t, X) &= \pi_s \lambda \hat{P}(t, T, \hat{\Psi}_t, \lambda^{-1} X) + \pi_r \hat{P}(t, T, \hat{\Psi}_t, X) + O(\mathrm{e}^{-(T-t)a}). \end{aligned} \tag{36}$$

We point out that, in view of the interpretation of the process Ψ_t for power prices with spikes in terms of the structural model that we discussed earlier in the article, the ergodic probabilities π_s and π_r can be interpreted as the unavailability and availability of a representative power plant. The unavailability, or the forced outage rate, of a power plant is the ratio of the forced outage hours to the total of the in service hours and the forced outage hours. Similarly, the availability of a power plant is the ratio of the in service hours to the total of the in service hours and the forced outage hours. In this way, the value $E(t, T, g)$ in (35) of a European contingent claim on power with spikes can be represented in terms of the weighted sum of the values $\hat{E}(t, T, g_{\lambda_T=\lambda})$ and $\hat{E}(t, T, g_{\lambda_T=1})$ of European contingent claims that expire during a spike and between spikes with weights equal to the unavailability and availability of a representative power plant.

We comment that the presented analysis in the special case of spikes with constant magnitude λ and the time-homogeneous Markov process M_t can be extended to the general case [1], [2], [4]. For example, in the case of the time-independent conditional probability density function $\Xi(\lambda)$ relationships (32–35) are still valid if $g_{\lambda_T=\lambda}$ is replaced by the conditional average $\bar{g}$ of the payoff g_{λ_T} over the magnitude of spikes λ_T at time T given in (27)

Finally, we consider [2] the following practically important special case when the risk-neutral dynamics of power prices $\hat{\Psi}_t$ in the absence of spikes are given by a geometric mean-reverting process defined in (12). In this case, relationship (34) takes the following form

$$E(t, T, g)(\hat{\Psi}_t) = \hat{E}^{MR}(t, T, \bar{g}_{\mathrm{erg}})(\hat{\Psi}_t) + O(\mathrm{e}^{-(T-t)a}), \tag{37}$$

where the value of the European contingent claim in the mean-reverting market environment can be found with the help of relationship (17). Moreover [2], in the case under consideration the relationships in (36) take the following form

$$\begin{aligned} C(t, T, \hat{\Psi}_t, X) &= \pi_s \lambda \hat{C}^{MR}(t, T, \hat{\Psi}_t, \lambda^{-1}X) + \pi_r \hat{C}^{MR}(t, T, \hat{\Psi}_t, X) + O(\mathrm{e}^{-(T-t)a}), \\ P(t, T, \hat{\Psi}_t, X) &= \pi_s \lambda \hat{P}^{MR}(t, T, \hat{\Psi}_t, \lambda^{-1}X) + \pi_r \hat{P}^{MR}(t, T, \hat{\Psi}_t, X) + O(\mathrm{e}^{-(T-t)a}), \end{aligned} \tag{38}$$

where the values of the European call and put options in the mean-reverting market environment can be found with the help of relationships (20) and (21). We comment that in view of relationships (19) and (18) the value in (37) of a European contingent claim on power with spikes can be represented in terms of the value of a European contingent claim in the Black-Scholes market environment. For example, in view of relationship (20) the values in (38) of European call and put options on power with spikes can be represented in terms of the values of European call and put options in the Black-Scholes market environment that are given by the Black-Scholes formulas in (21).

4. Modeling power forward prices for power with spikes

In this section we present an analytical expression for the power forward prices in the case when the dynamics of the power prices, or more precisely power spot prices, with spikes are described by the non-Markovian process Ψ_t defined earlier in the article. This is assuming that the corresponding power forward prices are known in the case of the process $\hat{\Psi}_t$ for power prices in the absence of spikes. We also show how the power forward prices far enough from the maturity time of the forward contracts on power do not exhibit spikes while the power prices do. This allows for the analytical valuation and dynamic hedging of European contingent claims on forwards on power in the practically important special case when $\hat{\Psi}_t$ is a geometric mean-reverting process.

4.1. Power forward prices for power without spikes

In order to find power forward prices in the case of the power prices, or more precisely power spot prices, with spikes following the non-Markovian process Ψ_t we need to find the corresponding power forward prices in the case of the process $\hat{\Psi}_t$ for power prices in the absence of spikes.

Assume now that $\hat{\Psi}_t$ is a risk-neutral process for power prices in the absence of spikes. Then the power forward price $\hat{F}(t, T) = \hat{F}(t, T)(\hat{\Psi}_t)$ at time t for the forward contract with maturity time T can be found as the risk-neutral expected value of the power prices $\hat{\Psi}_T$ at time T:

$$\hat{F}(t, T)(\hat{\Psi}_t) = \int_0^\infty P(t, T, \hat{\Psi}_t, \hat{\Psi}_T)\hat{\Psi}_T \mathrm{d}\hat{\Psi}_T, \tag{39}$$

where $P(t, T, \hat{\Psi}_t, \hat{\Psi}_T)$ is the risk-neutral transition probability density function for $\hat{\Psi}_t$ as a Markov process.

For example, in the practically important special case when the risk-neutral dynamics of power prices $\hat{\Psi}_t$ in the absence of spikes are given by a geometric mean-reverting process defined in (12), relationship (39) takes the following form [10], [2], [3], [13]:

$$\hat{F}(t, T)(\hat{\Psi}_t) = \mathrm{e}^{(T-t)\frac{1}{2}\hat{\sigma}^2(t,T)}\mathrm{e}^{b(t,T)}\hat{\Psi}_t^{a(t,T)}, \tag{40}$$

where $\hat{\sigma}(t, T)$, $a(t, T)$ and $b(t, T)$ are given in (17). Moreover, the risk-neutral dynamics of the power forward prices $\hat{F}(t, T)$ are given by the geometric Brownian motion with the zero drift:

$$\mathrm{d}\hat{F}(t, T) = \sigma_{\hat{F}}(t)\hat{F}(t, T)\mathrm{d}W_t, \tag{41}$$

where the volatility $\sigma_{\hat{F}}(t) > 0$ is given by

$$\sigma_{\hat{F}}(t) = \sigma(t)\mathrm{e}^{-\int_t^T \eta(\tau)\mathrm{d}\tau}.$$

4.2. POWER FORWARD PRICES FOR POWER WITH SPIKES

Now we will present an analytical expression for the power forward prices in the case when the dynamics of the power prices, or more precisely power spot prices, with spikes are described by the non-Markovian process Ψ_t.

As we discussed earlier in the article the state of the power market at any time $t \geq 0$ can be fully characterized by any pair of the values of the processes Ψ_t, λ_t and $\hat{\Psi}_t$ at time t. In this regard, the power forward price $F(t, T)$ at time t for the forward contract (on power with spikes) with maturity time T can be represented as a function of any pair of the values of the processes Ψ_t, λ_t,and $\hat{\Psi}_t$ at time t. Throughout this article we will represent $F(t, T)$ as a function of λ_t and $\hat{\Psi}_t$, that is $F(t, T) = F_{\lambda_t}(t, T)(\hat{\Psi}_t) = F(t, T)(\lambda_t\hat{\Psi}_t, \lambda_t)$. We comment that the power forward price $F(t, T)$ at time t for the forward contract (on power with spikes) with maturity time T is either $F_{\lambda_t}(t, T)(\hat{\Psi}_t) = F(t, T)(\lambda_t\hat{\Psi}_t, \lambda_t)$ units of account if the power price $\Psi_t = \lambda_t\hat{\Psi}_t$ is in the spike state at time t and the magnitude of the spike is $\lambda_t > 1$, or $F_{\lambda_t=1}(t, T)(\hat{\Psi}_t) = F(t, T)(\lambda_t\hat{\Psi}_t, \lambda_t)|_{\lambda_t=1}$ units of account if the power price $\Psi_t = \lambda_t\hat{\Psi}_t$ is in the inter-spike state at time t, that is $\lambda_t = 1$.

As we also discussed earlier in the article, the non-Markovian process Ψ_t can be represented as a Markov process with the state space that at any time $t \geq 0$ consists of all possible pairs $(\lambda_t, \hat{\Psi}_t)$ with $\lambda_t \geq 1$ and $\hat{\Psi}_t > 0$. Assume now that Ψ_t is a risk-neutral process for power prices with spikes so that it can be represented as a risk-neutral Markov process with the state space that at any time $t \geq 0$ consists of all possible pairs $(\lambda_t, \hat{\Psi}_t)$ with $\lambda_t \geq 1$ and $\hat{\Psi}_t > 0$. Then the power forward price $F(t, T)$ at time t for the forward contract (on power with spikes) with maturity time T can be found as the risk-neutral expected value of the power prices Ψ_T at time T. Moreover, it can be shown [2], [3] with the help of relationships (10) and (39) that the power forward price $F(t, T)$ for power with spikes can be expressed in terms of the corresponding power forward price $\hat{F}(t, T)$ for power without spikes:

$$F_{\lambda_t}(t, T)(\hat{\Psi}_t) = \bar{\lambda}_{\lambda_t}(t, T)\hat{F}(t, T)(\hat{\Psi}_t), \tag{42}$$

where $\bar{\lambda}_{\lambda_t}(t, T)$ is the risk-neutral *average magnitude of spikes* at time T given by

$$\bar{\lambda}_{\lambda_t}(t, T) = \begin{cases} P_{ss}^s(T, t)\lambda_t + \int_1^\infty \Xi_s(t, T, \lambda_T)\lambda_T\mathrm{d}\lambda_T + P_{rs}(T, t) & \text{if } \lambda_t > 1 \\ \int_1^\infty \Xi_r(t, T, \lambda_T)\lambda_T\mathrm{d}\lambda_T + P_{rr}(T, t) & \text{if } \lambda_t = 1 \end{cases}, \tag{43}$$

where $P_{ss}^s(T, t)$ is given in (8), and $\Xi_s(t, T, \lambda_T)$ and $\Xi_r(t, T, \lambda_T)$ are given in (25).

In a special case when the conditional probability density function $\Xi(t, \lambda)$ for the mutiplicative magnitude of spikes that start at time t is independent of time t and is equal to $\Xi(\lambda)$, relationship (43) takes the following form

$$\bar{\lambda}_{\lambda_t}(t, T) = \begin{cases} P_{ss}^s(T, t)\lambda_t + P_{ss}^r(T, t)\bar{\lambda} + P_{rs}(T, t) & \text{if } \lambda_t > 1 \\ P_{sr}(T, t)\bar{\lambda} + P_{rr}(T, t) & \text{if } \lambda_t = 1 \end{cases}, \tag{44}$$

where $\bar{\lambda}$ is the risk-neutral *conditional average magnitude of spikes* given by

$$\bar{\lambda} = \int_1^{\infty} \Xi(\lambda)\lambda d\lambda, \tag{45}$$

and where $P^r_{ss}(T,t)$ is given in (9).

Moreover, in a special case of spikes with constant magnitude λ, that is, when the conditional probability density function $\Xi(\lambda')$ is the Dirac delta function $\delta(\lambda - \lambda')$, relationship (44) and hence (43) take the following form

$$\bar{\lambda}_{\lambda_t}(t,T) = \begin{cases} P_{ss}(T,t)\lambda + P_{rs}(T,t) & \text{if } \lambda_t = \lambda \\ P_{sr}(T,t)\lambda + P_{rr}(T,t) & \text{if } \lambda_t = 1 \end{cases}. \tag{46}$$

Finally, we comment that it is with the help of the expression in (42) for the power forward prices $F(t,T)$ for power with spikes that the risk-neutral process Ψ_t for power prices with spikes can be calibrated to market data.

4.3. Why power forward prices do not exhibit spikes while power prices do

We will show [2], [3] that when the time to maturity of a forward contract on power is relatively large with respect to the duration of spikes the power forward prices do not exhibit spikes while the power prices, or more precisely power spot prices, do.

For simplicity, we again consider a special case when the spikes have constant magnitude λ and the Markov process M_t is time-homogeneous. In this case the power forward price $F(t,T)$ at time t for the forward contract with maturity time T is determined by relationship (42) with the average magnitude of spikes $\bar{\lambda}_{\lambda_t}(t,T)$ given by (46), and the transition matrix $P(T,t) = P(T-t)$ of the Markov process M_t is given by (5).

In view of relationship (30) the average magnitudes of spikes $\bar{\lambda}_{\lambda_t=\lambda}(t,T)$ and $\bar{\lambda}_{\lambda_t=1}(t,T)$ given by (46) coincide up to the terms of the order $O(\mathrm{e}^{-(T-t)a})$:

$$\bar{\lambda}_{\lambda_t=\lambda}(t,T) = \bar{\lambda}_{erg} + O(\mathrm{e}^{-(T-t)a}) \quad \text{and} \quad \bar{\lambda}_{\lambda_t=1}(t,T) = \bar{\lambda}_{erg} + O(\mathrm{e}^{-(T-t)a}), \tag{47}$$

where $\bar{\lambda}_{\mathrm{erg}}$ is the risk-neutral *ergodic average magnitude of spikes* given by

$$\bar{\lambda}_{\mathrm{erg}} = \pi_s\lambda + \pi_r, \tag{48}$$

with the ergodic probabilities π_s and π_r given in (31).

In turn, the power forward prices $F_{\lambda_t=\lambda}(t,T)(\hat{\Psi}_t)$ and $F_{\lambda_t=1}(t,T)(\hat{\Psi}_t)$ determined by relationship (42) with the average magnitude of spikes $\bar{\lambda}_{\lambda_t=\lambda}(t,T)$ and $\bar{\lambda}_{\lambda_t=1}(t,T)$ given by (47) also coincide up to the terms of the order $O(\mathrm{e}^{-(T-t)a})$ and hence can be combined into a single expression as follows:

$$F(t,T)(\hat{\Psi}_t) = \bar{\lambda}_{erg}\hat{F}(t,T)(\hat{\Psi}_t) + O(\mathrm{e}^{-(T-t)a}), \tag{49}$$

where $F(t,T)(\hat{\Psi}_t)$ is either $F_{\lambda_t=\lambda}(t,T)(\hat{\Psi}_t)$ or $F_{\lambda_t=1}(t,T)(\hat{\Psi}_t)$.

In this regard, if the time to maturity $T-t$ of the forward contract on power is relatively large with respect to the expected lifetime of a spike $\bar{t}_s = 1/a$ then the power forward prices $F_{\lambda_t=\lambda}(t,T)(\hat{\Psi}_t)$ and $F_{\lambda_t=1}(t,T)(\hat{\Psi}_t)$ during a spike and between spikes differ only by an exponentially small term of the order $O(\mathrm{e}^{-(T-t)a})$. As a result, the power forward prices do not exhibit spikes while the power prices do. In turn, the power forward prices can start exhibiting spikes and resembling the power prices when the forward contract nears its maturity with the time to maturity $T-t$ approaching the expected lifetime of a spike $\bar{t}_s = 1/a$. For example, consider a practically important special case when, in the absence of spikes in power prices, the risk-neutral dynamics of power forward prices $\hat{F}(t, T)$ are given by the geometric Brownian motion in (41). This is, for example, when the risk-neutral dynamics of power prices $\hat{\Psi}_t$ in the absence of spikes are given by a geometric mean-reverting process in (12) so that $\hat{F}(t, T)$ is given in (40). Then, even in the presence of spikes in the power prices, when the time to maturity $T - t$ of the forward contract on power is relatively large with respect to the expected lifetime of a spike $\bar{t}_s = 1/a$ the risk-neutral dynamics of power forward prices $F(t, T)$ are also given by the geometric Brownian motion in (41). This can be used to estimate the volatility of the geometric Brownian motion for the power forward prices $\hat{F}(t, T)$, as well as the volatility and the mean-reversion rate of the geometric mean-reverting process for the power prices $\hat{\Psi}_t$. This can also be used in the valuation and dynamic hedging of European contingent claims on forwards on power that we will discuss later in the article.

Finally, we comment that the presented analysis in the special case of spikes with constant magnitude λ and the time-homogeneous Markov process M_t can be extended to the general case [2], [3]. For example, in the case of the time-independent conditional probability density function $\Xi(\lambda)$ relationships (47–49) are still valid if the magnitude of spikes λ is replaced by the conditional average magnitude of spikes $\bar{\lambda}$ given by (45).

4.4. VALUATION AND HEDGING OF EUROPEAN CONTINGENT CLAIMS ON FORWARDS ON POWER FOR POWER WITH SPIKES

Consider a practically important special case when the risk-neutral dynamics of power prices $\hat{\Psi}_t$ in the absence of spikes are given by a geometric mean-reverting process in (12). We will show that the value of a European contingent claim on forwards on power for power with spikes can be expressed in terms of the value of a European contingent claim on forwards on power in the Black-Scholes market environment. This allows for the dynamic hedging of European contingent claims on forwards on power for power with spikes via the standard delta hedging.

For simplicity we again consider a special case when the spikes have constant magnitude λ and the Markov process M_t is time-homogeneous. It can be shown [2] that due to relationships (19), (40) and (49) the value $E(t, T, g)$ in (37) of a European contingent clam on power with spikes can be expressed in terms of the power forward price $F(t, T)$ for power with spikes:

$$E(t, T, g)(F) = \hat{E}^{BS}_{\hat{\sigma}(t,T),0}(t, T, \bar{g}_{\mathrm{erg}})(F/\bar{\lambda}_{\mathrm{erg}}) + O(\mathrm{e}^{-(T-t)a}). \tag{50}$$

Moreover, it can be shown [2] that the value $E(t, T, g)$ in (50) of a European contingent claim on forwards on power for power with spikes can be expressed in terms of the value of a European contingent claim on forwards on power in the Black-Scholes market environment:

$$E(t, T, g)(F) = \hat{E}^{BS}_{\hat{\sigma}(t,T),0}(t, T, g_{\mathrm{erg}})(F) + O(\mathrm{e}^{-(T-t)a}), \tag{51}$$

where $g_{\mathrm{erg}}(\hat{\Psi}_T) = \bar{g}_{\mathrm{erg}}(\hat{\Psi}_T/\bar{\lambda}_{\mathrm{erg}})$, and where we used the homogeneity of the values of European contingent claims in the Black-Scholes market environment.

In the case under consideration, as we discussed earlier in the article, if the time to maturity $T-t$ of the forward contract on power is sufficiently large with respect to the expected lifetime of a spike $\bar{t}_s = 1/a$ then the risk-neutral dynamics of power forward prices $F(t, T)$ for power with spikes are given by the geometric Brownian motion in (41). Therefore, relationship (51) allows for the dynamic hedging of European contingent claims on forwards on power for power with spikes via the standard delta hedging.

Finally, it can be shown [2] that in the case of the European call and put options with strike X relationship (51) takes the following form:

$$\begin{aligned}C(t,T,F,X) = \pi_s(\lambda/\bar{\lambda}_{\mathrm{erg}})\hat{C}^{BS}_{\hat{\sigma}(t,T),0}(t,T,F,(\lambda/\bar{\lambda}_{\mathrm{erg}})^{-1}X) + \\ \pi_r(1/\bar{\lambda}_{\mathrm{erg}})\hat{C}^{BS}_{\hat{\sigma}(t,T),0}(t,T,F,\bar{\lambda}_{\mathrm{erg}}X) + O(\mathrm{e}^{-(T-t)a}), \\ P(t,T,F,X) = \pi_s(\lambda/\bar{\lambda}_{\mathrm{erg}})\hat{P}^{BS}_{\hat{\sigma}(t,T),0}(t,T,F,(\lambda/\bar{\lambda}_{\mathrm{erg}})^{-1}X) + \\ \pi_r(1/\bar{\lambda}_{\mathrm{erg}})\hat{P}^{BS}_{\hat{\sigma}(t,T),0}(t,T,F,\bar{\lambda}_{\mathrm{erg}}X) + O(\mathrm{e}^{-(T-t)a}).\end{aligned}$$

5. Extensions of the model and conclusions

The model for power prices with spikes and for the valuation of European contingent claims on power presented in this article can be extended in the following ways:

- Both positive and negative multiplicative spikes as well as additive spikes and spikes of more complex shapes can also be considered [1], [2].
- The assumption that the spike process and the process for power prices in the absence of spikes are independent can be relaxed [1], [2]. Moreover, the spike process can be replaced with an arbitrary suitable Markov process, or even a finite product of arbitrary suitable Markov processes. This includes, as a special case, a multi-state Markov process and the product of several two-state Markov processes. For example, in view of the interpretetion of the non-Markovian process for power prices with spikes in terms of the structural model that we discussed earlier in the article, a multi-state Markov process can be used to model spikes in power prices as a result of partial outages and intermittent operation of a representative power plant. In turn, a product of several two-state Markov processes can be used to model spikes in power prices as a result of failure of multiple power plants.
- European contingent claims on power with spikes and other commodities whose prices do not exhibit spikes can also be valued. Those include fuel-sensitive, weather-sensitive and load-sensitive contingent claims such as spark spread options and full requirments contracts [25], [26].
- European contingent claims on power at several distinct points on the grid with spikes in all power prices can also be valued. Those include transmission options [27].
- Contingent claims of a general type such as universal contingent claims on power with spikes can also be valued with the help of the semilinear evolution equation for universal contingent claims [2], [16]. Those include Bermudan and American options. Universal contingent claims and the semilinear evolution equation for universal contingent claims were introduced by the author in [10], [18–22].

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